Electromagnetic tomography: method in time domain

Yozo Hamano

hamano@eps.s.u-tokyo.ac.jp Dept. Earth and Planetary Sci., University of Tokyo, Japan

Abstract. We present a new time-domain approach to the forward modeling of 3-D electromagnetic induction in a heterogeneous conducting sphere excited by external and internal souces. This method can be efficiently used to analyse global scale transient geomagnetic variations in order to infer the 3-D conductivity structures of the earth.

1 Introduction

The forward modeling of electromagnetic induction in a spherical earth with a 3D electrical conductivity structure has recently been actively investigated. These numerical calculations include a method based on the integral equation approach (Kuvshinov et al., 1999), a finite element method (Everett and Schulz, 1996), a finite difference method with staggerd grid (Uyeshima and Schulz, 1999), and an spectral-finite element approach (Martinec,1999). All of the methods for the global earth modeling calculate the induction in frequency domain in order to investigate a steady-state part of geomagnetic variations with a strictly periodic current system flowing in the Earth. Moreover, these studies require a large amount of computaition even for a single frequency and ,therefore, the inversion of the global earth structure from the observations is still under development. Since we are interested in the transient responses of the induction due to external magnetic disturbance and the variation of the geomagnetic field generated in the core, we deal with the induction problem in time domain. The purpose of the present paper is to develop the time-domain approach to compute a 3-D electromagnetic response of the earth to the external disturbances and the disturbances of core origin with various time scales.

2 Formulation of electromagnetic induction initial value problem

We solve the following Maxwell equations for B and E , simultaneously.

$$\frac{\partial B}{\partial t} = -\nabla \times (\eta \nabla \times B)$$
$$\frac{\partial E}{\partial t} = -\eta \nabla \times \nabla \times E$$

where the relation between E and B is expressed as $E = \eta \nabla \times B$, and the magnetic diffusivity, $\eta = 1/\sigma\mu_0$, varies as a function of position. Because the magnetic field, B, is solenoidal, B can be decomposed into two parts: toroidal and poloidal fields.

$$B = \nabla \times (Tr) + \nabla \times \nabla \times (\mathrm{Pr})$$

Substituting this relation into the above diffusion equations, and applying the angular momentum operator of quantum mechanics,

$$L^2 = -\left\{ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right\}$$

we get two equations for the poloidal, P, and toroidal T scaler functions in spherical coordinate.

$$\begin{aligned}
&L^2 \frac{\partial}{\partial t}[rP] = \eta L^2 \left(\frac{\partial^2}{\partial r^2} - \frac{L^2}{r^2} \right) [rP] \\
&- \left\{ \left(\frac{\partial\eta}{\partial\theta} \right) \frac{\partial}{\partial\theta} + \left(\frac{1}{\sin\theta} \frac{\partial\eta}{\partial\varphi} \right) \frac{1}{\sin\theta} \frac{\partial}{\partial\varphi} \right\} \left(\frac{\partial^2}{\partial r^2} - \frac{L^2}{r^2} \right) [rP] \\
&- \left\{ \left(\frac{\partial\eta}{\partial\theta} \right) \frac{1}{\sin\theta} \frac{\partial}{\partial\varphi} - \left(\frac{1}{\sin\theta} \frac{\partial\eta}{\partial\varphi} \right) \frac{\partial}{\partial\theta} \right\} \frac{\partial}{\partial r} [rT]
\end{aligned}$$

$$\begin{aligned}
&L^2 \frac{\partial}{\partial t}[rT] = \eta L^2 \left(\frac{\partial^2}{\partial r^2} - \frac{L^2}{r^2} \right) [rT] \\
&+ \left(\frac{\partial\eta}{\partial r} \right) L^2 \frac{\partial}{\partial r} [rT] \\
&+ \left(L^2 \eta \right) L^2 [rT] \\
&- \left\{ \left(\frac{\partial\eta}{\partial\theta} \right) \frac{\partial}{\partial\theta} + \left(\frac{1}{\sin\theta} \frac{\partial\eta}{\partial\varphi} \right) \frac{1}{\sin\theta} \frac{\partial}{\partial\varphi} \right\} \left(\frac{\partial^2}{\partial r^2} - \frac{2L^2}{r^2} \right) [rT] \\
&- \left\{ \left(\frac{\partial}{\partial\theta} \frac{\partial\eta}{\partial r} \right) \frac{\partial}{\partial\theta} + \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\varphi} \frac{\partial\eta}{\partial r} \right) \frac{1}{\sin\theta} \frac{\partial}{\partial\varphi} \right\} \left(\frac{\partial^2}{\partial r^2} - \frac{2L^2}{r^2} \right) \frac{\partial}{\partial r} [rT] \\
&+ \left\{ \left(\frac{\partial\eta}{\partial\theta} \right) \frac{1}{\sin\theta} \frac{\partial}{\partial\varphi} - \left(\frac{1}{\sin\theta} \frac{\partial\eta}{\partial\varphi} \right) \frac{\partial}{\partial\theta} \right\} \frac{\partial}{\partial r} \left(\frac{\partial^2}{\partial r^2} - \frac{L^2}{r^2} \right) [rP] \\
&+ \left\{ \left(\frac{\partial}{\partial\theta} \frac{\partial\eta}{\partial r} \right) \frac{1}{\sin\theta} \frac{\partial}{\partial\varphi} - \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\varphi} \frac{\partial\eta}{\partial r} \right) \frac{\partial}{\partial\theta} \right\} \left(\frac{\partial^2}{\partial r^2} - \frac{L^2}{r^2} \right) [rP]
\end{aligned}$$

It can be seen from these equations that the two modes (toroidal and poloidal) are mutually coupled, and the equations should be solved simultaneously.

To solve these equation, we expand the scalar fields, P, T, and η by the spherical harmonics.

$$
\begin{aligned}
&\eta\left(r,\theta,\varphi\right) \\
&= \eta^{*}\left[1+\sum_{n=1}^{\infty}\sum_{k=0}^{n}\left\{\eta_{n}^{kc}\cos k\varphi+\eta_{n}^{ks}\sin k\varphi\right\}P_{n}^{k}\left(\theta\right)\right] \\
&= \eta^{*}\left[1+\eta'\left(\theta,\varphi\right)\right] \\
&P\left(r,\theta,\varphi\right)=\sum_{n=1}^{\infty}\sum_{k=0}^{n}\left\{p_{l}^{mc}\cos k\varphi+p_{l}^{ms}\sin k\varphi\right\}P_{l}^{m}\left(\theta\right) \\
&T\left(r,\theta,\varphi\right)=\sum_{n=1}^{\infty}\sum_{k=0}^{n}\left\{t_{l}^{mc}\cos k\varphi+t_{l}^{ms}\sin k\varphi\right\}P_{l}^{m}\left(\theta\right)
\end{aligned}
$$

Since we are going to solve the 3-D induction problems of the spherical layered structure, we assume that the magnetic diffusivity varies laterarry and not a function of r within each layer. Substituting these equations into above, we get a set of diffusion equations for each coefficient of the poloidal and the toroidal field.

$$
\begin{aligned}
&\frac{1}{\eta^{*}}\frac{\partial}{\partial t}\left[rp_{l}^{mc,s}\right]-\left(\frac{\partial^{2}}{\partial r^{2}}-\frac{l(l+1)}{r^{2}}\right)\left[rp_{l}^{mc,s}\right]=\frac{1}{l(l+1)}f_{l}^{mc,s} \\
&\frac{1}{\eta^{*}}\frac{\partial}{\partial t}\left[rt_{l}^{mc,s}\right]-\left(\frac{\partial^{2}}{\partial r^{2}}-\frac{l(l+1)}{r^{2}}\right)\left[rt_{l}^{mc,s}\right]=\frac{1}{l(l+1)}g_{l}^{mc,s}
\end{aligned}
$$

These equations are well-known diffusion equations in spherical coordinates, and can be solved as an initial value problem, if the left-hand side of the equations can be explicitly expressed with the coefficients of the spherical harmonic expansion.As evident from eq.(4), the reduction involves decomposition of products of two Legendre function, two lateral derivatives of Legendre functions into sums of single Legendre functions. After executing these calculations, the right-hand side of the equations, f and g, can be expressed as follows,

$$
\begin{aligned}
&f_{l}^{mc,s}= \\
&\sum_{n,kc,s}\left(\Psi_{0}\right)_{n,kc,s}^{l,mc,s}n\left(n+1\right)\left(\frac{\partial^{2}}{\partial r^{2}}-\frac{n(n+1)}{r^{2}}\right)\left[rp_{n}^{kc,s}\right] \\
&-\sum_{n,kc,s}\left(\Psi_{+}\right)_{n,kc,s}^{l,mc,s}\left(\frac{\partial^{2}}{\partial r^{2}}-\frac{n(n+1)}{r^{2}}\right)\left[rp_{n}^{kc,s}\right] \\
&-\sum_{n,kc,s}\left(\Psi_{-}\right)_{n,kc,s}^{l,mc,s}\frac{\partial}{\partial r}\left[rt_{n}^{kc,s}\right]
\end{aligned}
$$

$$
\begin{aligned}
&g_{l}^{mc,s}= \\
&\sum_{n,kc,s}\left(\Psi_{0}\right)_{n,kc,s}^{l,kc,s}n\left(n+1\right)\left(\frac{\partial^{2}}{\partial r^{2}}-\frac{n(n+1)}{r^{2}}\right)\left[rt_{n}^{kc,s}\right] \\
&-\sum_{n,kc,s}\left(\Psi_{+}\right)_{n,kc,s}^{l,kc,s}\left(\frac{\partial^{2}}{\partial r^{2}}-\frac{2n(n+1)}{r^{2}}\right)\left[rt_{n}^{kc,s}\right] \\
&-\sum_{n,kc,s}l\left(l+1\right)\left(\Psi_{0}\right)_{n,kc,s}^{l,kc,s}\frac{n(n+1)}{r^{2}}\left[rt_{n}^{kc,s}\right] \\
&-\sum_{n,kc,s}\left(\Psi_{-}\right)_{n,kc,s}^{l,kc,s}\frac{\partial}{\partial r}\left(\frac{\partial^{2}}{\partial r^{2}}-\frac{n(n+1)}{r^{2}}\right)\left[rp_{n}^{kc,s}\right]
\end{aligned}
$$

where three operators,Ψ_{o}, Ψ_{+} *and* Ψ_{-},are defined as

$$
\begin{aligned}
&\left(\eta' P\right)_{l}^{mc,s}= \\
&\Psi_{o}P=\sum_{n,kc,s}\left(\Psi_{o}\right)_{n,kc,s}^{l,mc,s}P_{n}^{kc,s} \\
&\left\{\left(\frac{\partial\eta}{\partial\theta}\right)\frac{\partial}{\partial\theta}+\left(\frac{1}{\sin\theta}\frac{\partial\eta}{\partial\varphi}\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right\}P= \\
&\Psi_{+}P=\sum_{n,kc,s}\left(\Psi_{+}\right)_{n,kc,s}^{l,mc,s}P_{n}^{kc,s} \\
&\left\{\left(\frac{\partial\eta}{\partial\theta}\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}-\left(\frac{1}{\sin\theta}\frac{\partial\eta}{\partial\varphi}\right)\frac{\partial}{\partial\theta}\right\}P= \\
&\Psi_{-}P=\sum_{n,kc,s}\left(\Psi_{-}\right)_{n,kc,s}^{l,mc,s}P_{n}^{kc,s}
\end{aligned}
$$

Since the above three operators are only the functions of the structure, and not depend on the magnetic field, the operators can be calculated in matrix form before the time integral of the diffusion equations start. During the time integral, multiplication of the operater matrix with the radial derivative of the expansion coefficients give the left-hand side of the diffusion equations.

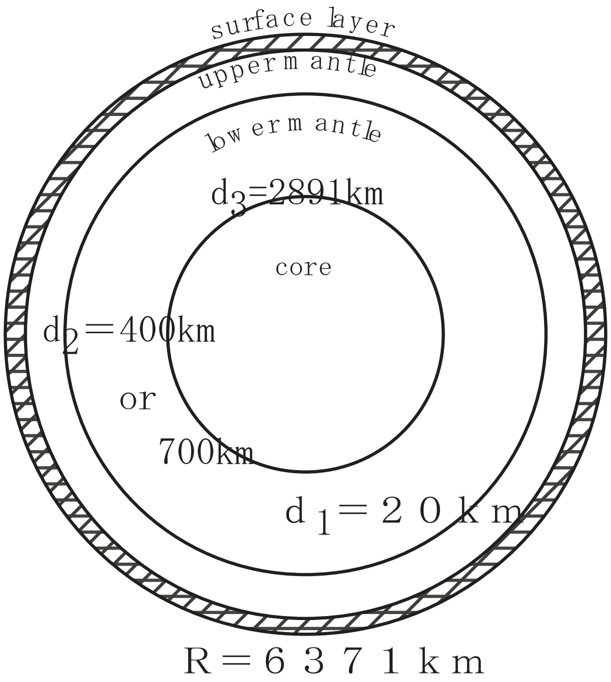

Fig. 1. Representation of a simple 4-layear earth model. Surface layer(Layer 1) has a lateral heterogeneity. 2nd (layer2) and 3rd layer correspond to the upper and the lower mantle,respectively.

3 A simple earth model with a surface heterogeneous layer

In order to verify the present time-domain approach to the induction problem of 3-D earth, we calculate the induction of a simple 4-layer model of the earth shown in Fig.1. The surface thin layer with a thickness of 20km is considered to be a simplification of the surface distribution of the ocean and the continents. The electrical resistivity in this layer is laterally heterogeneous with the distribution of

$$\rho_1 = 1 + P_1^1(\theta)\cos\varphi(\Omega m)$$

The second layer is laterally homogeneous with the resistivity of 100 ohm-m. Two models with the depths of the bottom of the second layer of 400 km (Model 400) and 700 km (Model 700) are considered to estimate the effect of the depth of this boundary. The third layer extend to the core-mantle boundary (2891 km depth), and the resistiviy of the layer is fixed to be 1 ohm-m. Core is assumed to be a perfect conducter, and all the coefficients are set to zero at this boundary.

The time integration started when the external field of

$$q_1^0(\theta,\varphi) = P_1^0(\theta)$$

is suddenly applied at t=0. Time derivatives of the induced field generated by this abrupt change of the external field give the impulse responses of the induced field. and the response function in frequency domain can be obtained by the Fourier transform of the impulse responses.

For the time integral of the diffusion equations, we approximate the time and the space derivatives with finite differences. Then, if we denote $u_j^n = rp_l^{mc,s}(t_n, r_j)$, the finite difference equation becomes

$$\begin{aligned}&-D\tfrac{\Delta t}{2\Delta x^2}u_{j-1}^{n+1}+\\&\left(1+D\tfrac{\Delta t}{\Delta x^2}+D\tfrac{l(l+1)\Delta t}{2(1+(j-1)\Delta x)^2}\right)u_j^{n+1}\\&-D\tfrac{\Delta t}{2\Delta x^2}u_{j+1}^{n+1}\\&=D\tfrac{\Delta t}{2\Delta x^2}u_{j-1}^{n}\\&+\left(1-D\tfrac{\Delta t}{\Delta x^2}-D\tfrac{l(l+1)\Delta t}{2(1+(j-1)\Delta x)^2}\right)u_j^{n}\\&+D\tfrac{\Delta t}{2\Delta x^2}u_{j+1}^{n}+D\tfrac{\Delta t}{l(l+1)}f^n\end{aligned}$$

where we used Crank-Nicholson Schema for the diffusion term of the equation for the stability of the computation. For $u_j^n = rt_l^{mc,s}(t_n, r_j)$, a similar equation is obtained. Boundary conditions at the internal boundaries are generally expressed as

$$\langle p_l^{mc,s}\rangle = \langle t_l^{mc,s}\rangle = \left\langle \frac{dp_l^{mc,s}}{dr}\right\rangle = \left\langle \eta\frac{dt_l^{mc,s}}{dr}\right\rangle = 0$$

where$\langle ... \rangle$ denotes the jump across the boundary. Finite difference approximation of these boundary conditions are

$$\begin{aligned}&-\tfrac{\Delta x_2}{\Delta x_1+\Delta x_2}u_{jb-1}^{n+1}+u_{jb}^{n+1}\\&-\tfrac{\Delta x_1}{\Delta x_1+\Delta x_2}u_{jb+1}^{n+1}=0\end{aligned}$$

or

$$\begin{aligned}&-\tfrac{D_1\Delta x_2}{D_2\Delta x_1+D_1\Delta x_2}u_{jb-1}^{n+1}+u_{jb}^{n+1}\\&-\tfrac{D_2\Delta x_1}{D_2\Delta x_1+D_1\Delta x_2}u_{jb+1}^{n+1}\\&=-\tfrac{D_1\Delta x_2}{D_2\Delta x_1+D_1\Delta x_2}\eta'\left(u_{jb}^{n}-u_{jb-1}^{n}\right)\end{aligned}$$

for $u = rp_l^{mc,s}$ and $u = rt_l^{mc,s}$, respectively. And the surface boundary condition is given by

$$\left(l+\frac{r}{\Delta x_1}\right)u_1^{n+1}-\frac{r}{\Delta x_1}u_2^{n+1} = -\frac{2l+1}{l+1}v_1^{n+1}$$

where $v = rq_l^m$ denotes the coefficients of the external field at the surface.

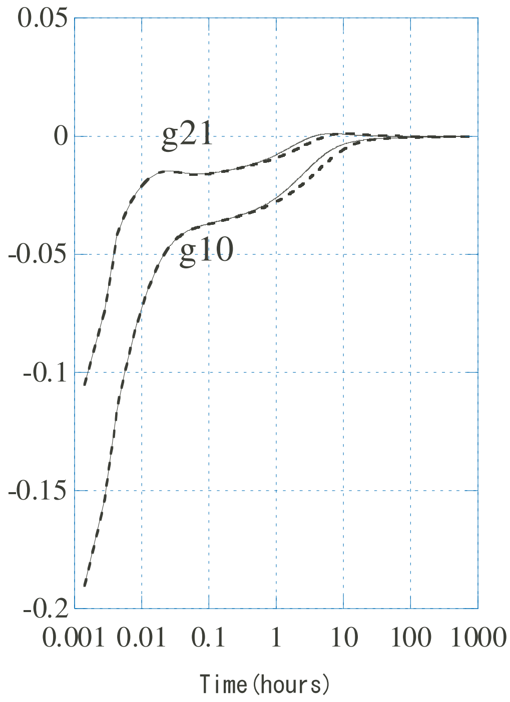

Fig. 2. Impulse response of the internal gauss coefficients g10 and g21 to the external disturbance of q10. Solid line:Model400, Dotted line:Model700.

In the present study, time variations of the expansion coefficients up to degree 10 were calculated. Time step of the calculation is 5 second, and the integration was continued until 730 hours (about a month). The calulated responce of the induced field was differentiated with time to obtain the impulse response of the simplified earth model. Among the induced field, g10 and g21 components are primaly induced phase within the surface layer. The behavior of the time variation of these components are shown in Fig.2, where the unit of the response functions is normalized to the applied external field. Initial inductions of these components are large and rapidly decay with time. The change of the decay rate corresponds to the radial conductivity structure of the earth. In Fig.2, difference of the two models (Model400 and Model 700) can be detected after an hour has elapsed, suggesting the possibility to estimate the depth of the conductivity jump at around this time scale. Compared to these primary induced phase, time variations of the secondary induced phases are quite different. Dominant components of these secondary phase are zonal fields of g30,g50,g70,.... and g32,g41,g43,g52,g61,g63,g65,g72,g74,.... etc. These components are generated by the interactions between the primary and the secondary induced phases, and the structure. An example of the time behaviour of these secondary induced components are shown in Fig.3. These secondary phases have double peaks at around 0.001-0.01 hours and around 1-20 hours. The induced current for the first peak can be attributed to the current flow within the surface heterogeneous layer and the top of the upper mantle, since the response functions for Model400 and Model700 are identical. On the other hand, the current system generating the second peak flows into the lower mantle and returns to the surface. This time-scale might be important to determine the depth and the topography of the upper and the lower mantle boundary.

Behaviour of the response function of the present simple earth can be more clearly understood when we see the spatial distribution of the induced field. In Fig.4, vertical component, Bz, of the induced field on the surface are shown, where the conductive ocean hemisphere of the present model spans from latitude 90 to 270, and the other hemisphere is a resistive continental hemisphere. In the figure, the field due to g10 component is removed so that we can see the secondary induced phase. The initially induced field consists of g10 and g20, and is symetric for the two hemispheres. These field rapidly decreases with time, but the decay rate is more gradual in the ocean hemisphere. Hence, after an hour, the induced field in the ocean hemisphere becomes larger than that of the continental hemisphere. This tendency increases with time, and after 4 hours, the return current from the lower mantle causes new anomalies at high latitudes in the ocean hemisphere. These anomalies are in opposist polarity with the initially induced field, and they grows with time and move to the lower latitude (Fig.4c). After 10 hours, these anomalies start to decrease with time, but the anomaly remains even after 30 days. Since all the presently observed induced fields are originally induced in the surface heterogeneous layer with a thickness of only 20 km, it is to be noted that this surface heterogeneous layer should be included in the model even if the purpose of the investigation is the middle and the lower mantle. In addition, the difference of the response functions of the present two models suggests that these suface induced field at the time scale of 1 to 100 hours is useful to estimate the conductivity structure of the middle mantle.

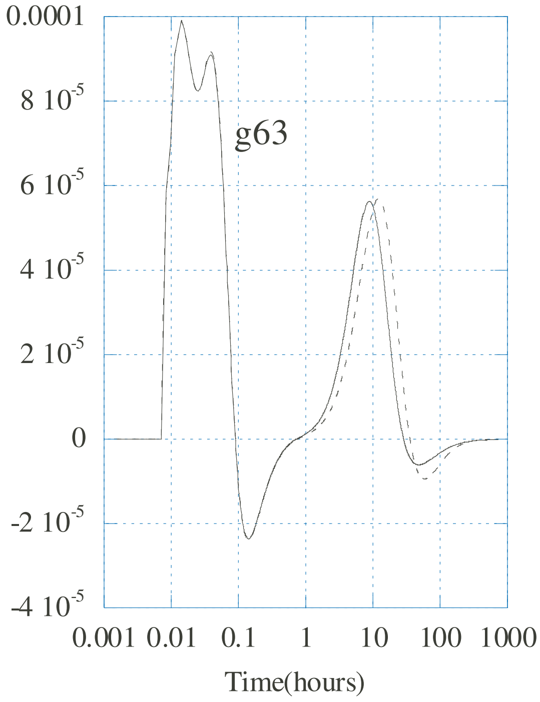

Fig. 3. Impulse response of the internal gauss coefficients g63.

4 Conclusion

The theory of elctromagnetic induction within the three-dimensional spherical earth has been developed in time domain. The method permits conductivity heterogeneities inside the earth to calculate the response of the earth to the external and internal geomagnetic disturabances. This method is powerful to analyse the transient phenomena related to those disturabances.

5 Reference

Everett,M.E. and A. Schultz, Geomagnetic induction in a heterogeneous sphere: azimuthally symmetric test computations and the response of an undulating 660-km discontinuity, *J. geophys. Res.*, **101**,2765–2783, 1996.

Kuvshinov, A.V.,D.B. Avdeev, and O.V. Pankratov, Global induction by Sq and Dst sources in the presence of oceans: bimodal solutions for non-uniform spherical surface shells above radially symmetric earth models in comparison to observations, *Geophys. J. Int.*, **137**, 630–650, 1999.

Martinec, Z., Spectral-finite element approach to three-dimensional electromangetic induction in a spherical earth, *Geophys. J. Int.*, **136**, 229–250, 1999.

Uyeshima, M., and A. Schultz, Geoelectromanetic induction in a heterogeneous sphere: a new three-dimensional forward solver using a conservative staggered-grid finite difference method, *Geophys. J. Int.*, **140**, 636–50, 1999.

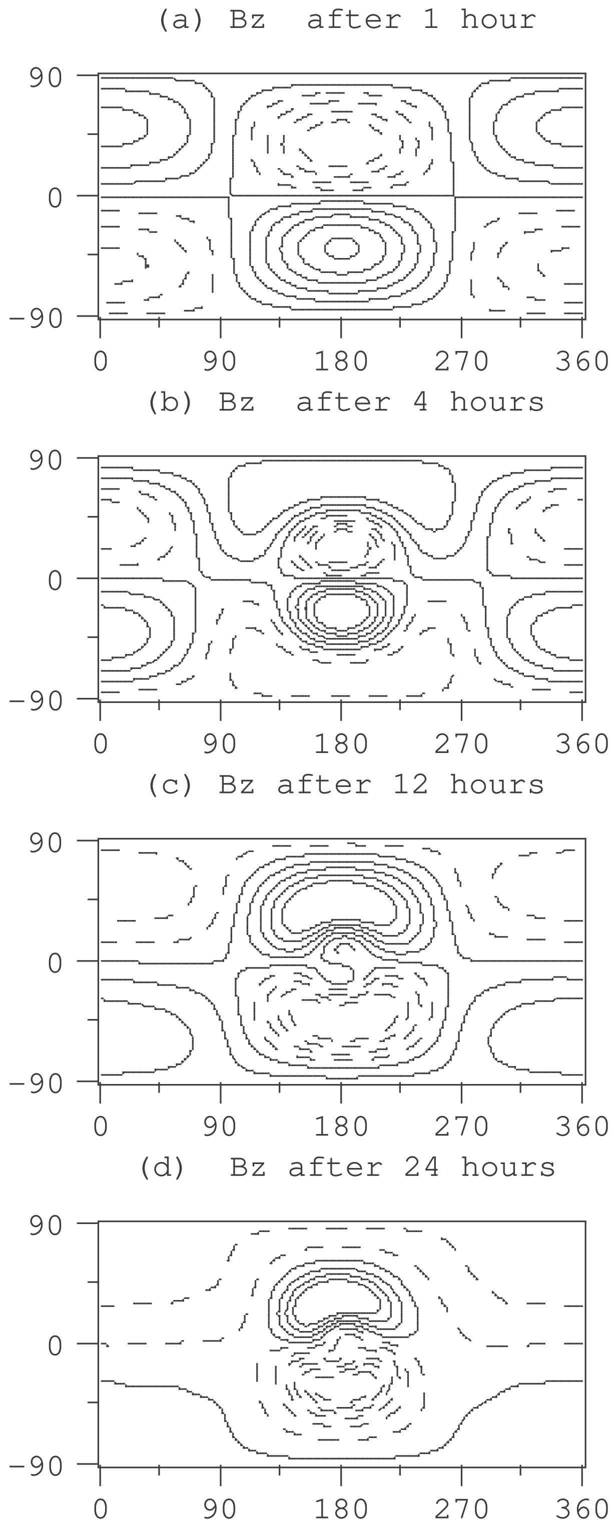

Fig. 4. Spatial distribution of the vertical components of the induced field. (a)after 1 hour,(b)after 4 hours, (c) after 12 hours, (d) after 1day.